The Open University

Mathematics: A Second Level Course

Linear Mathematics M201
Bridging Material 1

DIFFERENTIAL EQUATIONS—An Introduction

Prepared by the Course Team

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Contents

Set Books

D. L. Kreider, R. G. Kuller, D. R. Ostberg and F. W. Perkins, *An Introduction to Linear Analysis* (Addison-Wesley, 1966).
E. D. Nering, *Linear Algebra and Matrix Theory* (John Wiley, 1970).

It is essential to have these books; the course is based on them and will not make sense without them.

Conventions

Before working through this correspondence text make sure you have read *An Introduction to the Bridging Material* and *A Guide to the Linear Mathematics Course.*

The set books are referred to as:

K for *An Introduction to Linear Analysis*
N for *Linear Algebra and Matrix Theory*

Note

This bridging material is not based on the set books. It has been written especially for the benefit of students who have taken the Mathematics Foundation Course M101 (The Open University Press, 1978)
References to this foundation course take the form M101 Block V *Unit 2*

1.0 INTRODUCTION

Many problems in applied mathematics require, for a complete solution, the specification of a *function.* As a relatively straightforward example, imagine that you have a coil spring, one end of which is fixed, and that you suspend from the other end a weight (such as £1's worth of copper coins!) Imagine further that you pull down on the weight, and then release it. During the subsequent motion, the weight will bob up and down: the problem is to predict as accurately as possible, from a knowledge of characteristics of the spring and the weight, and of the physical laws which apply to such a system, the motion of the weight. This motion can best be described by a function, the function which gives the distance of the weight, below a suitable fixed point on the axis of the spring, in terms of the time elapsed since the weight's release.

In this case, as in those dealt with in M101 Block V *Unit 2*, the information from which the unknown function is to be found consists of a *differential equation* that it has to satisfy. That is, the information takes the form of a relation that must hold between one (or several) of the function's *derivatives*, and the function itself. To be explicit, the distance function must have the property that its second derived function, when added to the function itself, gives zero, i.e.

$$f'' + f = 0$$

where f is the distance function.

That is to say, for every time t (after the moment of release) we must have

$$f''(t) + f(t) = 0.$$

(This differential equation is a consequence of the law governing the motion of a body which says that the body's acceleration is proportional to the force acting on it. The acceleration at time t is $f''(t)$; the force in this case happens to be proportional to $f(t)$; when the constants of proportionality take suitable values, and the direction in which the force acts is taken into account, the differential equation above is the result. The details of the derivation need not concern us.)

Differential equations arise in many different situations. One activity which very often leads to a differential equation is the *modelling* of a *time-dependent system*, where a rule governing the rate of change of some quantity is known, and the problem is to determine the quantity's dependence on time explicitly. Dynamical problems like the spring problem described above lead to differential equations because the law of motion determines only acceleration. Some purely geometrical problems lead to differential equations—for example, one might want to determine all curves whose tangents had some special property.

1.1 SOLUTIONS OF DIFFERENTIAL EQUATIONS

1.1.1 First steps

Having established the importance of differential equations, we turn now to begin to study them in their own right. To start with, here are some more examples of differential equations:

$$F'(t) = 2t \tag{1}$$

$$U'(t) = -\lambda U(t) \tag{2}$$

$$\frac{du}{dt} = -\lambda u \tag{3}$$

$$\frac{dy}{dx} = y^2 \tag{4}$$

$$\frac{d^2y}{dx^2} + y = 0 \tag{5}$$

$$2x\frac{dy}{dx} - 3y = 3x^3. \tag{6}$$

Here are some points to note about these examples.

First, there is a lot of latitude in the notation we use for differential equations. We can use *primes* to denote derivatives, as in examples (1) and (2), or we can use *Leibniz' notation* as in the other examples.

In fact, in example (5) we have simplified the expression $\frac{d^2}{dx^2}(y)$, for the second derivative of y, to the form $\frac{d^2y}{dx^2}$. Similarly we shall denote the mth derivative by $\frac{d^my}{dx^m}$. We shall often refer to the derived function as the derivative, for brevity, as does the set book **K**.

The same differential equation may appear in quite different guises, as in examples (2) and (3). We can even change letters without changing the essential meaning of a differential equation: as you have probably realized, the list of examples includes the equation that introduced this discussion. For the remainder of this section we shall use the form of writing differential equations of which (4), (5) and (6) are examples.

As a second point, note that the differential equation may include not only the unknown function and its derivatives, but also the 'dependent variable'—either by itself (the $2t$ in example (1), the $3x^3$ in example (6)) or multiplying a derivative (the $2x$ in example (6)).

Third, note that example (5) differs from all the others in that it involves the second derivative while the others only involve the first. We describe this difference by calling examples (1)–(4) and (6) *first-order* differential equations, and example (5) a *second-order* differential equation. The order of a differential equation is the order of the highest derivative that occurs in it. Thus

$$x^2\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} + y = 0$$

is a second-order equation,

$$\frac{d^4y}{dx^4} + \frac{d^2y}{dx^2} + \sin x = 0$$

is a fourth-order one. The same terminology is used even when the derivative occurs in some more complicated fashion:

$$\left(\frac{dy}{dx}\right)^2 + 8y = e^x$$

is a first-order differential equation, even though the derivative is squared. (Equations like this, however, will not appear very often in this course.)

The main point of studying differential equations is to *solve* them. To solve a differential equation is to find a function (or functions) of x, which, when it is substituted for y (and its derivatives are substituted for $\frac{dy}{dx}, \frac{d^2y}{dx^2}, \ldots$ as necessary) causes the equation to be satisfied. It is occasionally possible to do this by *guesswork*: for example, a knowledge of the derivatives of the sine and cosine functions immediately suggest that

$y = \sin x$, (whose derivative is discussed in M101 Block III *Unit 1*, Section 1.2)

is a possible solution of the differential equation

$$\frac{d^2y}{dx^2} + y = 0.$$

It is a straightforward matter to check that it is a *solution*:

$$\text{if } y = \sin x, \text{ then } \frac{dy}{dx} = \cos x, \text{ and } \frac{d^2y}{dx^2} = -\sin x,$$

so

$$\frac{d^2y}{dx^2} + y = (-\sin x) + \sin x = 0.$$

It is equally easy to check that

$$y = \cos x$$

is also a solution of this equation—and that so are

$$y = \sin x + \cos x$$

$$y = 2 \sin x$$

$$y = 2 \sin x + 3 \cos x$$

and, indeed, so is

$$y = a \sin x + b \cos x$$

for *any* constants a and b.

This is the first indication of a general truth; that a differential equation may have *many* solutions—and by that we don't mean tens or hundreds, we mean so many that one cannot count them. *Each* choice of the values of the numbers a and b in the formula above gives a solution of the differential equation. As it happens, every solution of the equation can be obtained by making an appropriate choice of a and b. So in this case, at least, we can be said to have solved the equation completely by the formula

$$y = a \sin x + b \cos x.$$

There are many interesting types of differential equation for which a

complete solution, in this sense, can be found. It will be our aim, in this course, to describe some of them and their solutions.

Guesswork, the "method" we used above, is clearly not the most satisfactory method of solving differential equations. (If you don't believe this, try guessing a solution of say

$$2x\frac{dy}{dx} - 3y = 3x^3!)$$

Systematic methods would be preferable, and we shall describe some below, and later in the course. But we should first make it clear that there is no one systematic method for solving *all* differential equations (as there is one systematic method for solving all quadratic equations in algebra, for example). Instead, one has to pick out certain types of differential equation, and develop a separate method for each type.

The simplest type of differential equation is exemplified by the equation

$$\frac{dy}{dx} = 2x$$

(compare with example (1) above!). Here the derivative of the unknown function is simply specified, as a function of x. In other words, the differential equation says that the unknown function is a *primitive* of $2x$. Solution of this differential equation amounts to antidifferentiation, and we obtain

$$y = x^2 + c$$

where c is a constant. Note that putting in the *constant of integration*, c, is again a way of admitting that there are many solutions to the equation.

You should find it easy to recognize differential equations of this simple type—and to solve them, at least if you can carry out the integration. The point of mentioning this type of equation is that in the last analysis, integration is the only method we have for finding explicit solutions of differential equations. All other methods are really methods of getting the equation into a form in which integration will lead to the required solution. (This ignores, of course, the possibility of finding approximate solutions by graphical or computational methods—a most important branch of the subject, but not one we are immediately concerned with.)

From this point of view we can understand, albeit very crudely, where those constants a and b in the solution

$$y = a \sin x + b \cos x$$

of the differential equation

$$\frac{d^2y}{dx^2} + y = 0$$

come from: they are in some sense 'constants of integration', and there are two of them because it takes two integrations to undo the two differentiations built into this second-order equation.

Before going on to describe some methods of solving some more complicated first-order differential equation, let us summarize the story so far.

A differential equation is an equation involving an unknown function, its derivatives, and the dependent variable. Such equations are of frequent occurrence in applications of mathematics. The order of a differential equation is the order of the highest derivative it contains. A solution of

a differential equation is a function which satisfies the equation: when the function's derivatives are computed and substituted into the equation, together with the function itself if necessary, the equation is satisfied. It cannot be emphasized too strongly that the solution of a differential equation is a function. The simplest kind of differential equation is one in which the unknown function's derivative is specified in terms of x. Such an equation is solved by direct integration. Other differential equations must also be solved by integration, but indirectly.

Exercises

1. Solve the differential equation

$$\frac{dy}{dx} = \sin x + x.$$

2. Find the solution of the equation

$$\frac{d^2y}{dx^2} + y = 0$$

that also satisfies $y(0) = 1$ and $y\left(\frac{\pi}{2}\right) = 1$.

Solutions

1. The function $y = -\cos x + \frac{x^2}{2} + c$ is a primitive for $\sin x + x$, where c is an arbitrary constant of integration. Hence this function y is a solution of the given differential equation.

2. A typical solution of the equation is

$$y = a \sin x + b \cos x,$$

where a, b are constants.

Now $y(0) = a.0 + b.1 = b$

and $y\left(\frac{\pi}{2}\right) = a.1 + b.0 = a$

So the solution

$$y = \sin x + \cos x$$

is obtained by setting $a = b = 1$.

1.1.2 Separation of Variables

We begin by reviewing a method of solving certain first-order differential equations which was discussed in M101 Block V *Unit 2*. It is called the method of *separation of variables*. We start with a simple example. Consider the equation

$$\frac{dy}{dx} + \lambda y = 0, \qquad \text{where } \lambda \text{ is a constant.}$$

Assuming that y is not zero for any value of x, we may rewrite this equation as

$$\frac{1}{y}\frac{dy}{dx} = -\lambda.$$

The left hand side of this equation is in fact a derivative of a function of y, the $\frac{dy}{dx}$ term arising from the application of the *chain rule*:

$$\frac{d}{dx}(\log_e y) = \frac{1}{y}\frac{dy}{dx}.$$

(The differentiation of $\log_e$ was discussed in M101 Block III, *Unit 5* Section 5.4.)

Thus the equation may be written as

$$\frac{d}{dx}(\log_e y) = -\lambda$$

and its solution reduces to a very straightforward integration:

$$\log_e y = \int(-\lambda)dx = -\lambda x + c, \qquad (c \text{ is a constant}),$$

whence

$$y = e^{-\lambda x + c} = ae^{-\lambda x},$$

where $a = e^c$ is another constant. (Strictly speaking if $a = e^c$ then a must apparently be positive. However a more careful analysis shows that our method implicitly assumes that y is always positive; it may easily be modified to include the negative solutions, which take the same form but with a negative; the zero function is also a solution, and this is included in the general solution $y = ae^{-\lambda x}$ as the case $a = 0$.)

Before continuing with differential equations we must make a brief aside here about logs. In the calculations above we have used the notation $\log_e$ for the natural logarithm function (log to base e). This was the notation used in M101. However, in M201 the alternative notation ln is used for this function (because it is used in **K**); so we shall adopt this notation from now on.

The method we used for solving the differential equation $\frac{dy}{dx} + \lambda y = 0$ may be adapted for use in quite a large class of first order equations. The equations to which it applies are those of the form

$$\frac{dy}{dx} + f(x)\,g(y) = 0,$$

where f and g are any two functions. In our example above these functions were particularly simple:

$$f(x) = \lambda \qquad g(y) = y.$$

But the method would also apply to

$$\frac{dy}{dx} - y^2 = 0,$$

$$\frac{dy}{dx} - \frac{x}{y} = 0,$$

$$\frac{dy}{dx} + \sin x \sin y = 0,$$

for example. It would not however apply to

$$\frac{dy}{dx} + \sin(xy) = 0.$$

In the general case

$$\frac{dy}{dx} + f(x)\,g(y) = 0$$

it works like this. We *separate the variables* by rewriting the equation in the form

$$\frac{1}{g(y)}\frac{dy}{dx} = -f(x).$$

The left-hand side does not involve x explicitly, the right hand side involves x alone. The variables can thus be said to be separated.

In the particular case we described above we were able to express the left hand side as the derivative of a function of y, that is, as the derivative of a composite function, the $\frac{dy}{dx}$ term coming from the chain rule. To do this in the general case, we have to find a primitive of the left hand side, that is, evaluate the indefinite integral

$$\int \frac{1}{g(y)}\frac{dy}{dx}\,dx.$$

But by the *substitution rule*, (M101 Block III *Unit 3*, Section 3.5) this is the same as

$$\int \frac{1}{g(y)}\,dy,$$

which can be evaluated (in principle, at any rate) since g is a known function. We are then in the happy position of having an equation of the form

$$\frac{d}{dx}(\text{something}) = -f(x),$$

which can be solved by a further integration. It remains only to solve the resulting equation algebraically for y.

In fact the method in practice is very simple: to find the solution of

$$\frac{dy}{dx} + f(x)\,g(y) = 0,$$

one simply rewrites it as

$$\int \frac{1}{g(y)}\,dy = \int (-f(x))\,dx,$$

evaluates the integrals, and solves the result algebraically for y.

Example

Solve the equation

$$\frac{dy}{dx} - \frac{x}{y} = 0.$$

Solution

On separation of the variables, the equation becomes

$$y\frac{dy}{dx} = x.$$

This is equivalent to

$$\int y\,dy = \int x\,dx$$

so that

$$\frac{1}{2}y^2 = \frac{1}{2}x^2 + c$$

or

$$y = \sqrt{x^2 + d}, \qquad \text{where } d \text{ is a constant.}$$

(Of course, $d = 2c$—but it makes no difference, in picking out a particular solution, whether we have to specify the value of d or of $2c$.)

It is worth emphasizing here that it is always a good habit to check the correctness of a solution of a differential equation by substitution in the equation. In this case,

$$\text{if } y = \sqrt{x^2 + d}, \text{ then } \frac{dy}{dx} = \frac{1}{2}\cdot\frac{1}{\sqrt{x^2 + d}}\cdot 2x = \frac{x}{\sqrt{x^2 + d}},$$

and so

$$\frac{dy}{dx} - \frac{x}{y} = \frac{x}{\sqrt{x^2 + d}} - \frac{x}{\sqrt{x^2 + d}} = 0.$$

So we have succeeded in producing a solution.

Exercises

1. Solve the equation

$$\frac{dy}{dx} - \frac{y}{x} = 0,$$

(assuming x and y are always positive).

2. Find a solution of

$$\frac{dy}{dx} - \frac{1}{y} = 0,$$

for which

$$y(1) = \sqrt{2}.$$

Solutions

1. Separating the variables we find

$$\int \frac{dy}{y} = \int \frac{dx}{x},$$

so integration gives

$$\ln y = \ln x + c, \qquad \text{for some constant } c.$$

Using properties of ln we find

$$y = kx,$$

where k is the constant such that $\ln k = c$.

2. Separating the variables we find

$$\int y\,dy = \int dx,$$

so integration gives

$$\tfrac{1}{2}y^2 = x + c, \qquad \text{for some constant } c.$$

Hence

$$y = \sqrt{2x + 2c}$$

and since we require that $y(1) = \sqrt{2}$ the solution we seek is

$$y = \sqrt{2x}.$$

1.1.3 Integrating Factor Method

An important case of a separable variables equation is of the form

$$\frac{dy}{dx} + P(x)y = 0, \tag{1}$$

where $P(x)$ is a specified function. To solve this equation, we separate the variables, to obtain

$$\int \frac{dy}{y} = -\int P(x)dx + A$$

i.e.

$$\ln|y| = -\int P(x)dx + A$$

or $\quad y = Ce^{-\int P(x)dx}, \qquad (\text{where } \ln|C| = A).$

So that a solution to (1) is $C = ye^{\int P(x)dx}$.

Why did the modulus sign appear in part of our solution? Well, the function $\ln x$ is only defined for positive value of x, so if y takes negative values the function $\ln y$ is not defined. However, in this case $\ln(-y)$ is defined, since $(-y)$ takes positive values. In either case, $\ln|y|$ is always defined since the modulus of a number is always positive.

Now if y is positive we know

$$\frac{d}{dx}(\ln|y|) = \frac{d}{dx}(\ln y) = \frac{1}{y}\frac{dy}{dx}.$$

But also if y is negative we find

$$\frac{d}{dx}(\ln|y|) = \frac{d}{dx}(\ln(-y)) = \frac{1}{-y}\left(-\frac{dy}{dx}\right) = \frac{1}{y}\frac{dy}{dx}.$$

Hence $\ln|y|$ is always a primitive for $\dfrac{1}{y}\dfrac{dy}{dx}$ and

$$\int \frac{1}{y}\frac{dy}{dx}dx = \int \frac{dy}{y} = \ln|y|.$$

Fortunately at the next stage in the argument the sign of y is incorporated into the constant C and so we can dispense with the modulus $|y|$.

Equation (1) is a special case of the equation

$$\frac{dy}{dx} + P(x)y = Q(x), \tag{2}$$

for which $Q(x) = 0$.

If $Q(x) \neq 0$ is a known function, we can multiply (2) by an *integrating factor*; it is

$$e^{\int P(x)dx}.$$

We do this so that the left-hand side of the differential equation then becomes a derived function of a product.

Indeed (2) becomes

$$e^{\int P(x)dx}\frac{dy}{dx} + e^{\int P(x)dx}P(x)y = e^{\int P(x)dx}Q(x),$$

which can be written as

$$\frac{d}{dx}\left(ye^{\int P(x)dx}\right) = e^{\int P(x)dx}Q(x)$$

Integrating, we obtain

$$ye^{\int P(x)dx} = \int e^{\int P(x)dx}Q(x)dx + C$$

or

$$y = \left(\int e^{\int P(x)dx}Q(x)dx + C\right)\cdot e^{-\int P(x)dx}$$

This is a very complicated expression, so here are some examples.

Example

Solve $2\frac{dy}{dx} + 4\frac{y}{x} = 8x$, given that $y = 2$ when $x = 1$.

Solution

This equation will be in the form (2) if the coefficient of $\left(\frac{dy}{dx}\right)$ is 1. So on dividing by 2, we can write the equation as

$$\frac{dy}{dx} + 2\frac{y}{x} = 4x.$$

Now $P(x) = \frac{2}{x}$ and $Q(x) = 4x$ and $\int P(x)dx = \ln x^2$.

The integrating factor is

$$e^{\int P(x)dx} = e^{\ln x^2} = x^2.$$

Multiplying the equation by the integrating factor, we obtain

$$x^2\frac{dy}{dx} + 2yx = 4x^3,$$

i.e.

$$\frac{d}{dx}(x^2y) = 4x^3.$$

On integrating, we find

$$x^2y = x^4 + C.$$

Since $y = 2$ when $x = 1$, we find

$$2 = 1 + C, \text{ and hence } C = 1.$$

The required solution is

$$y = \frac{1}{x^2} + x^2.$$

Example

Solve the equation

$$y' \tan x + y = 4 \sin x, \qquad \text{for } x \in]0, \pi[$$

given that $y = 2\sqrt{2}$ when $x = \dfrac{\pi}{4}$.

Solution

We write the equation in the standard form as

$$y' + y \cot x = 4 \cos x,$$

so that

$$\int P(x)dx = \int \cot x \, dx = \int \frac{\cos x}{\sin x} \, dx$$

$$= \ln|\sin x|$$

Since $\sin x$ is positive in the range we are considering we can drop the modulus sign. The integrating factor is therefore $\sin x$. On multiplication, the equation is written as

$$\frac{d}{dx}(y \sin x) = 4 \cos x \sin x = 2 \sin 2x$$

so that

$$y \sin x = 2 \int \sin 2x \, dx + C$$

$$= -\cos 2x + C = 2 \sin^2 x - 1 + C;$$

hence

$$y = 2 \sin x + \frac{C-1}{\sin x}.$$

Since $y = 2\sqrt{2}$ when $x = \dfrac{\pi}{4}$, we have $C = 2$.

It follows that $y = 2 \sin x + \dfrac{1}{\sin x}$.

Exercises

1. Solve the differential equation

$$\frac{1}{2}\frac{dy}{dx} - 3y = -x,$$

given that when $x = 0$, $y = \frac{1}{3}$.

2. Solve the equation

$$xy' + y = x \sin x$$

given that when $x = \dfrac{\pi}{2}$, $y = 1$.

Solution

1. The equation can be written as

$$\frac{dy}{dx} - 6y = -2x$$

The integrating factor is

$$e^{\int P(x)dx} = e^{-6\int dx} = e^{-6x}$$

Then

$$\frac{d}{dx}(ye^{-6x}) = -2xe^{-6x},$$

so that

$$ye^{-6x} = -2\int xe^{-6x} + C$$
$$= -2\left(-\frac{1}{6}xe^{-6x} - \frac{1}{36}e^{-6x}\right) + C$$

(on integrating by parts) and

$$y = \frac{1}{3}\left(x + \frac{1}{6}\right) + Ce^{6x}.$$

When $x = 0$, $y = \frac{1}{3}$ i.e.

$$\frac{1}{3} = \frac{1}{3} \times \frac{1}{6} + C \quad \text{or} \quad C = \frac{5}{18}.$$

Finally,

$$y = \frac{1}{3}\left(x + \frac{1}{6}\right) + \frac{5}{18}e^{6x}$$
$$= \frac{1}{3}\left(x + \frac{1}{6} + \frac{5}{6}e^{6x}\right).$$

2. We write the equation as

$$y' + \frac{y}{x} = \sin x.$$

The integrating factor is $e^{\int P(x)dx} = e^{\ln x} = x \ (x > 0)$.

Hence

$$\frac{d}{dx}(xy) = x \sin x$$

so that

$$xy = \int x \sin x \, dx = -x \cos x + \sin x + C.$$

Since $y = 1$ when $x = \frac{\pi}{2}$, we have

$$\frac{\pi}{2} = 1 + C$$

which implies that $C = \frac{\pi}{2} - 1$.

The solution is

$$y = \frac{1}{x}\left(\sin x + \frac{\pi}{2} - 1\right) - \cos x.$$

1.1.4 Summary of Sections 1.0 and 1.1

In these sections we defined the terms

differential equation	(page 5)
order of a differential equation	(page 6)
separation of variables	(page 9)
integrating factor	(page 13)

We introduced the notation

$$\frac{d^2y}{dx^2}, \frac{d^my}{dx^m} \qquad \text{(page 6)}$$

$\ln x$ (page 10)

We discussed four techniques for solving differential equations. They are:

1. *Guesswork* (page 7)

For example, $y = \sin x$ is a solution of

$$\frac{d^2y}{dx^2} + y = 0$$

2. *Integration* (page 8)

An equation of the form

$$\frac{dy}{dx} = f(x)$$

can be solved by finding a primitive for f.

3. *Separation of Variables* (page 11)

An equation of the form

$$\frac{dy}{dx} + f(x)g(y) = 0$$

can be solved by this method.

4. *Integrating factor* (page 13)

This method can be used to solve equations of the form

$$\frac{dy}{dx} + P(x)y = Q(x), \text{ using the integrating factor } e^{\int P(x)dx}.$$

1.2 DIFFERENTIAL OPERATORS

You may very well be wondering at this stage what this work on differential equations has to do with *vector spaces* and *linear transformations*. Some hints that there is a connection have already occurred. One is the reference to spaces of functions at various places in the early part of the course—as we have been at pains to point out, differential equations have very much to do with *functions* Another is that the general solution of the equation

$$\frac{d^2y}{dx^2} + y = 0$$

could be described as the set of all *linear combinations* of the functions $\sin x$ and $\cos x$: in other words it is a two-dimensional space of functions, which is *spanned* by $\sin x$ and $\cos x$. The set of solutions of

$$\frac{dy}{dx} + \lambda y = 0$$

is even easier to describe: it is the one-dimensional subspace of the space of differentiable functions, which is spanned by $e^{-\lambda x}$. The sets of solutions of these differential equations are both *vector spaces* (and it is no coincidence that the dimension of the solution space matches the order of the equation in each case).

To explain in a little more detail what these connections are we shall have to make a slight detour, and go back and interpret the operation of differentiation from a new viewpoint. How does differentiation fit in with function spaces? First of all, one must get used to thinking of differentiation as an *operation*, which can be applied to any suitable function, and which results in another function. To emphasize this point it is helpful to adopt yet another notation, one which separates the symbol for differentiation from the symbol for a function to which it is applied. We use D to stand for differentiation, so that the derivative of a function f is written

$$Df.$$

You should think of this as analogous to a linear transformation acting on a vector. In fact D is a linear transformation: given any two differentiable functions f and g, and constants a and b,

$$D(af + bg) = aDf + bDg.$$

(This is just a smart way of saying something that you have known for ages—the sum rule and constant multiple rule for differentiation.)

The rules for *combining* linear transformations, which are discussed in early units of M201, apply to linear transformations of function spaces just as much as to those on finite-dimensional vector spaces. Thus, since multiplication by a constant is a linear transformation (indeed, multiplication by a fixed function is a linear transformation on a function space), we may form new linear transformations such as

$$D + \lambda,$$

$$D^2 + 1.$$

Here D^2 means DD—apply D twice in succession. Since these are linear transformations, their *kernels* are vector spaces. And what are their kernels? The kernel of a linear transformation is the set of vectors which

it maps to zero. The kernel of $D + \lambda$ is thus the set of functions f such that

$$(D + \lambda)f = 0$$

or

$$Df + \lambda f = 0.$$

But this is just the differential equation

$$\frac{dy}{dx} + \lambda y = 0$$

in disguise. In other words, solving the differential equation amounts to finding the kernel of a linear transformation. The same is true of the differential equation

$$\frac{d^2y}{dx^2} + y = 0:$$

the linear transformation in this case is $D^2 + 1$. This explains why, in these two cases, the set of solutions is a *vector space*—it is the kernel of a linear transformation.

Not all differential equations by any means, can be described in this way. But those that can are especially important, simply because the vector space association suggests powerful ways of dealing with them. Differential equations which arise from linear transformations formed from the D operator in this way are called *linear differential equations*—they are the subject of much enquiry in this course.

In these remarks we have been deliberately imprecise about exactly which function spaces are involved when one discusses the D operator. To put the whole matter on a firm footing, one must first of all clear up this point; and this is where *Unit 4* begins.

Summary of Section 1.2

This section defines

combinations of operators	(page 18)
linear differential equations	(page 19)

It introduces the notation

D	(page 18)
D^2	(page 18)
Df	(page 18)
$D + \lambda$	(page 19)
$D^2 + 1$	(page 19)

This section shows that the set of solutions of a differential equation can sometimes be considered as the kernel of a linear transformation. Hence they are vector subspaces, and some spanning subsets were exhibited.